4. Binomial Expansions

4.1. Pascal's Triangle

The expansion of $(a+x)^2$ is

$$(a+x)^2 = a^2 + 2ax + x^2$$

Hence,

$$(a+x)^3 = (a+x)(a+x)^2 = (a+x)(a^2+2ax+x^2)$$
$$= a^3 + (1+2)a^2x + (2+1)ax^2 + x^3 = a^3 + 3a^2x + 3ax^2 + x^3$$

Further,

$$(a+x)^4 = (a+x)(a+x)^4 = (a+x)(a^3+3a^2x+3ax^2+x^3)$$
$$= a^4 + (1+3)a^3x + (3+3)a^2x^2 + (3+1)ax^3 + x^4$$
$$= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

In general we see that the coefficients of $(a+x)^n$ come from the n-th row of **Pascal's Triangle**, in which each term is the sum of the two terms just above it.

row 0					1				
row 1				1		1			
row 2			1		2		1		
row 3		1		3		3		1	
row 4	1		4		6		4		1
		.		.		.		.	

EXAMPLE 4.1. Find the expansion of $(2x-y)^4$.

$$(2x-y)^4 = \big((2x)+(-y)\big)^4 = (2x)^4 + 4(2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$$
$$= 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4.$$

4.2. Factorials

To understand the coefficients in Pascal's triangle we need the **factorial** function

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1\,;$$

it is read 'n factorial'.
$1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$, ...
By convention $0! = 1$ (see below).

Q. Given four objects, say the letters A, B, C, D, how many different orders can you put them in? For example, some possible orders are $ABCD$, $DCBA$, $ABDC$.
A. The first letter can be any one of the four, say C.
The second letter can be any one of the remaining three, say A.
The third letter can either of the remaining two, say D.
The fourth letter must be the remaining one, B.

In all there are $4 \times 3 \times 2 \times 1 = 4! = 24$ possible orders.
In general, given n different objects there are $n!$ possible orders or **permutations**.

Example 4.2. How many ways are there of arranging the letters in the word $PASCAL$? We have 6 letters. If they were all different there would be 6! arrangements. However, there are two A's, which themselves can be arranged in 2! ways. Therefore the number of arrangements is $6!/2! = 360$.

4.3. Combinations

Suppose we have 5 different objects, say A, B, C, D, E. How many ways are there to choose two of them? For example, some possible choices are AB, AC, BC, ...
(The order you choose them in doesn't matter, so AB is the same as BA.)
The first can be any one of the five.
The second can be any one of the remaining four.
This gives 20. But we have double counted. We get AB and BA, but these are the same. That gives $20/2 = 10$.
They are AB, AC, AD, AE, BC, BD, BE, CD, CE, DE.
Given n objects, the number of ways to choose 2 is

$$\frac{n(n-1)}{2} = \frac{n(n-1)}{2!} .$$

With 3 objects, we can permute the chosen ones in 3! ways without altering the choice, so that the number of ways to choose 3 is

$$\frac{n(n-1)(n-2)}{3!} .$$

In general, given n objects, the number of ways to choose r of them is

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!} .$$

It is read 'nCr' or 'n choose r', and sometimes denoted ${}^nC_r$.
Note that $\binom{n}{r} = \binom{n}{n-r}$; this is because selecting r objects is the same as choosing which $n-r$ objects to leave out, so that the number of ways of choosing $n-r$ objects from n is the same as the number of ways of choosing r objects from n.
The convention that $0! = 1$ ensures that $\binom{n}{0} = \binom{n}{n} = 1$. There is exactly one way to choose 0 (i.e., none) of the objects; equivalently, there is exactly one way to choose all n of them.
Also $\binom{n}{1} = n$. There are n ways to choose 1 object.

Example 4.3. To do the lottery you need to choose 6 numbers out of 49. There are

$$\binom{49}{6} = \frac{49!}{6!\,43!} = 13983816$$

ways to do this. Therefore, the probability that a given ticket will win the jackpot is 1/13983816.

4.4. Binomial Theorem

$$(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots + \binom{n}{r}a^{n-r}x^r + \cdots + \binom{n}{n-1}ax^{n-1} + x^n$$

For example

$$\begin{aligned}(a+x)^{20} &= a^{20} + \binom{20}{1}a^{19}x + \binom{20}{2}a^{18}x^2 + \dots \\ &= a^{20} + 20a^{19}x + \frac{20\cdot 19}{2!}a^{18}x^2 + \dots \\ &= a^{20} + 20a^{19}x + 190a^{18}x^2 + \dots\end{aligned}$$

Proof. When you expand

$$(a+x)^n = (a+x)(a+x)\dots(a+x)$$

you get a big sum involving terms like

$$axaaaxxaxa\dots$$

which are a product of n factors, each either an x or an a.
The coefficient of $a^{n-r}x^r$ is the number of terms in the sum which involve exactly r x's.
There are $\binom{n}{r}$ choices of r x's so that the coeffient of $a^{n-r}x^r$ is $\binom{n}{r}$. □

EXAMPLE 4.4. Find the term in x^5y^8 in $(2x-y^2)^9$.
The general term is

$$\binom{9}{i}(2x)^{9-i}(-y^2)^i$$

The term we want is the one with $i=4$, so it is

$$\begin{aligned}\binom{9}{4}(2x)^5(-y^2)^4 &= \frac{9!}{4!.5!}\,2^5(-1)^4x^5y^8 \\ &= \frac{9.8.7.6}{4.3.2.1}\,32x^5y^8 \\ &= 4032x^5y^8.\end{aligned}$$

4.5. Binomial series

The binomial theorem is for n-th powers, where n is a positive integer. Indeed $\binom{n}{r}$ only makes sense in this case.
However, the right hand side of the formula

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

makes sense for any n.
The Binomial Series is the expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

which is valid for any number n, positive or negative, integer or fractional, provided that $-1 < x < 1$.

Special cases.

$$\frac{1}{1+x} = (1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \dots$$

so

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$$

also

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \dots$$

so

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots$$

These are valid for all x with $-1 < x < 1$.

Convergence For x close to 0, the binomial series gives a good approximation very quickly. Considering the series for $1/(1+x)^2$ with $x = 0.1$, we have,

LHS $= 1/(1+0.1)^2 = 1/1.1^2 = 0.8264463...$,
RHS $= 1 - 0.2 + 0.03 - 0.004 + 0.0005 - \dots$.

Number of terms	Sum
1	1
2	0.8
3	0.83
4	0.826
5	0.8265
10	0.82644628

For x not so close to 0, but still in the range $-1 < x < 1$ the series converges, but more slowly. For example, with $x = 0.6$, LHS $= 1/(1+0.6)^2 = 1/1.6^2 = 0.390625$
RHS $= 1 - 1.2 + 1.08 - 0.864 + 0.648 - \dots$.

Number of terms	Sum
1	1
2	-0.2
3	0.88
4	0.016
5	0.6640
10	0.35047168
25	0.39067053

For x outside the range $-1 < x < 1$, the series doesn't converge and so is useless.
For example, with $x = 2$, LHS $= 1/(1+2)^2 = 1/3^2 = 0.1111111....$
RHS $= 1 - 4 + 12 - 32 + 80 - \ldots$.

Number of terms	Sum
1	1
2	-3
3	9
4	-23
5	57
10	-3527
25	283348537

Other expansions To expand $1/(1+2x)$, for example, write it as

$$\frac{1}{1+2x} = (1+(2x))^{-1} = 1 - (2x) + (2x)^2 - (2x)^3 + \ldots$$
$$= 1 - 2x + 4x^2 - 8x^3 + \ldots$$

This is valid for $-1 < 2x < 1$, so for $-\frac{1}{2} < x < \frac{1}{2}$.
Two more:

$$\frac{1}{1-x} = (1+(-x))^{-1} == 1 - (-x) + (-x)^2 - (-x)^3 + (-x)^4 - \ldots$$
$$= 1 + x + x^2 + x^3 + x^4 + \ldots$$

and

$$\frac{1}{(1-x)^2} = (1+(-x))^{-2} = 1 - 2(-x) + 3(-x)^2 - 4(-x)^3 + \ldots$$
$$= 1 + 2x + 3x^2 + 4x^3 + \ldots.$$

Both of these are valid for $-1 < (-x) < 1$, so for $-1 < x < 1$.

4.6. **Worked examples**

EXAMPLE 4.5. Expand the following expressions.
$(1 + x^2)^5$, $(x + y)(x + 2y)^4$,

EXAMPLE 4.6. How many arrangements are there of the letters in each of the following words?
SPAIN, ENGLAND, AUSTRALIA, MOROCCO

EXAMPLE 4.7. Compute the following binomial coefficients:
$\binom{100}{0}$; $\binom{100}{1}$; $\binom{100}{2}$; $\binom{100}{3}$.

EXAMPLE 4.8. In bridge, a player is dealt 13 out of 52 cards.
How many possible bridge hands are there?
A Yarborough is a hand which contains no aces, kings, queens, jacks or 10s. How many possible Yarborough hands are there?
What is the probability that a bridge hand is a Yarborough.

EXAMPLE 4.9. Find the expansions of the following, up to the term in x^3. In each case, state the range of validity for x.
$(1 - x)^{1/3}$, $(1 + 2x)^{-2}$.

EXAMPLE 4.10. Find the expansion of $\sqrt{1+x}$ up to the term in x^2. By taking $x = 1/4$, use your expansion to find an approximation to $\sqrt{5}$, giving your answer as a fraction.

$$\begin{aligned}(1+x)^{1/2} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \dots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots\end{aligned}$$

Putting $x = 1/4$ gives

$$\begin{aligned}\sqrt{1+\frac{1}{4}} &\cong 1 + \frac{1}{2}\frac{1}{4} - \frac{1}{8}\left(\frac{1}{4}\right)^2 \\ &= \frac{128}{128} + \frac{16}{128} - \frac{1}{128} = \frac{143}{128}.\end{aligned}$$

Therefore

$$\sqrt{5} = 2\sqrt{\frac{5}{4}} = 2\sqrt{1+\frac{1}{4}} \cong \frac{143}{64}.$$

(In fact $\frac{143}{64} = 2.234\dots$ and $\sqrt{5} = 2.236\dots$)